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STRUCTURAL STABILITY FOR FLOWS ON THE TORUS WITH A CROSS-CAP

BY

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ABSTRACT. Let $\mathfrak{X}^r(\tilde{M})$, $r > 1$, denote the space of C^r-vector fields on the torus with a cross-cap $\tilde{M}$. We show that the Morse-Smale vector fields of $\mathfrak{X}^r(\tilde{M})$ are dense on it. We also give a simple proof that a C^0-flow on the Klein bottle cannot support a nontrivial ω-recurrent trajectory.

Introduction. We denote by $\mathfrak{X}^r(M)$, $r = 1, 2, 3, \ldots$, the space of C^r-vector fields (with the C^r-topology) on a compact, connected, boundaryless, C^∞, two-dimensional manifold M.

In [2] Peixoto defines certain elements of $\mathfrak{X}^r(M)$ of a very simple orbit structure, called Morse-Smale vector fields. We denote by Σ^r the subset of $\mathfrak{X}^r(M)$ formed by these elements.

In this paper we prove the following.

THEOREM 1. *Let $\tilde{M}$ be the torus with a cross-cap. $X \in \mathfrak{X}^r(\tilde{M})$ is structurally stable if and only if $X \in \Sigma^r$. Moreover, Σ^r is open and dense in $\mathfrak{X}^r(\tilde{M})$.*

As a by-product of this work, we give a simple proof of the following

THEOREM 2 [1]. *Every α- or ω-recurrent trajectory of a continuous flow on the Klein bottle K^2 is trivial.*

Our result is motivated by the following theorem, due to M. Peixoto [2].

(A) Let M be orientable. $X \in \mathfrak{X}^r(M)$ is structurally stable if and only if $X \in \Sigma^r$. Moreover, Σ^r is open and dense in $\mathfrak{X}^r(M)$.

This theorem is stated in [2] for M orientable or not. According to its proof, Theorem (A) is true for M orientable or not if, and only if, it is possible to give an affirmative answer to the following question.

(B) Let $X \in \mathfrak{X}^r(M)$ have a finite number of singularities, all hyperbolic. Is it possible, by an appropriate arbitrarily small C^r-perturbation of X, to get a new vector field with only trivial α- and ω-recurrent trajectories?

Peixoto's proof of (B) [2] is correct only in the orientable case. We give

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more details about this in proof of Theorem 1.

Since the continuous flows on the real projective plane and on the Klein bottle do not exhibit nontrivial α- and ω-recurrent trajectories [1], Theorem (A) still holds in these cases.

In this work, we prove that trajectories accumulating on an α- or ω-recurrent trajectory of a continuous flow on the torus with a cross-cap have essentially the same behavior as in the case of orientable manifolds. We can therefore give an affirmative answer to question (B) and to verify (A) in this case. When M is a nonorientable manifold of genus $\geqslant 4$, an answer to (B) is unknown.

1. Proof of Theorem 2.

(1) DEFINITIONS. Let φ: $\mathbf{R} \times M \to M$ be a continuous flow on M and $\gamma = \gamma(t)$ a trajectory of φ. We set:

(i) $\alpha(\gamma) = \{p \in M | p = \lim_{n\to+\infty}(\gamma(t_n)), t_n \to -\infty, t_n \in \mathbf{R}\}$;

(ii) $\omega(\gamma) = \{p \in M | p = \lim_{n\to+\infty}(\gamma(t_n)), t_n \to +\infty, t_n \in \mathbf{R}\}$;

(iii) If $\gamma \subset \omega(\gamma)$ ($\gamma \subset \alpha(\gamma)$), we say that γ is ω-recurrent (α-recurrent). A fixed point as well as a closed orbit of φ is called a trivial ω-recurrent (α-recurrent) trajectory.

We say that a simple closed curve $C \subset M$ (*a circle*) is *transversal* to φ iff there exists an $\varepsilon > 0$, such that the map $(t, p) \to \varphi(t, p)$ is a homeomorphism of $[-\varepsilon, \varepsilon] \times C$ onto the closure of an open neighborhood of C.

Let C be a transverse circle (segment) to φ. An arc of a trajectory of φ, say $\widehat{p_1p_2}$, is called a *C-arc* iff $\widehat{p_1p_2} \cap C = \{p_1, p_2\}$. We say that a C-arc $\widehat{p_1p_2}$, $p_1 \neq p_2$, is one-sided (two-sided) if the union of $\widehat{p_1p_2}$ with one of the arcs of C whose endpoints are p_1 and p_2 is a one-sided (two-sided) simple closed curve.

(2) LEMMA (PEIXOTO). *Let γ be an ω-recurrent trajectory of a continuous flow φ: $\mathbf{R} \times M \to M$. Then there exists a transverse circle to φ through a point of γ.*

PROOF. If $\widehat{pq}$ (perhaps $p = q = \widehat{pq}$) is an arc of trajectory of φ which is not a fixed point, then $\widehat{pq}$ can be enclosed in a flow box; that is, there exist a subset S of M containing p, which is homeomorphic to a nondegenerate closed interval, and an $\varepsilon > 0$ such that the map $(x, t) \to \varphi(s, t)$ is a homeomorphism of $[-\varepsilon, \varepsilon] \times S$ onto the closure of an open neighborhood of $\widehat{pq}$. In fact, for the case $p = q = \widehat{pq}$, see [3]. The case $p \neq q$ follows easily from this.

By means of this result we are going to construct a transverse circle to φ through a point of γ. Given $t \in \mathbf{R}$, let us denote by φ_t the map defined by $\varphi_t(\tilde{p}) = \varphi(t, \tilde{p}), \tilde{p} \in M$. Let Σ be a segment transversal to φ passing through some $p \in \gamma$. There exists an $\varepsilon > 0$ such that $A = \{\varphi_t(\Sigma) | -\varepsilon \leqslant t \leqslant \varepsilon\}$ is a flow box (see Figure 1a). Since $p \in \omega(\gamma)$, γ intersects Σ infinitely many times. Let γ be the minimum positive real number satisfying $q = \varphi_\tau(p) \in \Sigma$. Let

$\Omega \in \Sigma$ be a segment containing p and such that $\{\varphi_t(\Omega) | 0 \leq t \leq \tau\} = B$ is a flow box. Let $p_0 = \varphi_{-\varepsilon}(p)$ and $q_0 = \varphi_{-\varepsilon}(q)$. Let "$<$" denote an orientation on $\varphi_{-\varepsilon}(\Sigma)$. We will consider only the case $q_0 < p_0$.

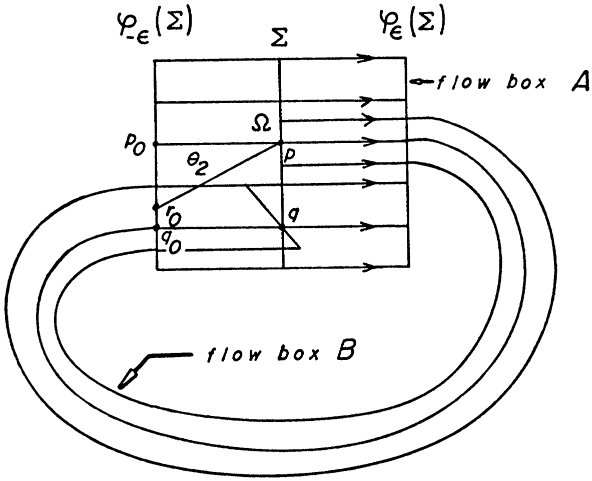

FIGURE 1a

Let $I = \{s \in \varphi_{-\varepsilon}(\Sigma) | q_0 \leq s \leq p_0\}$ and s_0 be the first point, after q_0, where γ intersects I. If the I-arc $\widehat{p_0 q_0}$ is two-sided we choose $r_0 \in I \cap B$ such that $q_0 < r_0 < p_0$. We denote by θ_1 (resp. θ_2) a segment contained in (B) (resp. in (A)) transversal to X and joining r_0 and p. See Figures 1a and 1b. Certainly we can select θ_1 and θ_2 satisfying $\theta_1 \cap \theta_2 = \{r_0, p\}$. Under this condition, since $\widehat{p_0 q_0}$ is two-sided, $\theta_1 \cup \theta_2$ is a transverse circle to φ. We remark that if $\widehat{p_0 q_0}$ were one-sided $\theta_1 \cup \theta_2$ would not be a transverse circle to φ.

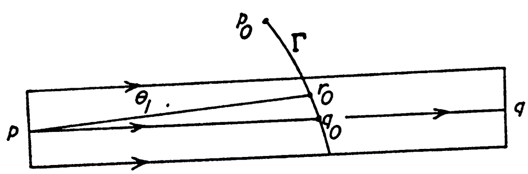

FIGURE 1b

The case where $\widehat{q_0 s_0}$ is two-sided is similar to the first one.

If $\widehat{p_0 q_0}$ and $\widehat{q_0 s_0}$ are one-sided, $\widehat{p_0 s_0}$ is a two-sided J-arc, being $J = \{s | s_0 \leq s \leq p_0\}$, and the transverse circle is constructed in a similar way.

(3) DEFINITIONS. Let Γ be a simple closed curve. Let M_1 and M_2 be the connected components of $M - \Gamma$.

If Γ is one-sided and therefore $M_1 = M_2$ we define $A_x(M_1)$ as the compactification of M_1 by means of a point x at infinity. If Γ is two-sided and M_1 and M_2 are disjoint, we define two disjoint manifolds $A_x(M_1)$ and $A_y(M_2)$ as above. Finally if Γ is two-sided and $M_1 = M_2$ we define $A_{xy}(M_1)$ as the compactification of M_1 by means of two distinct points, x and y, at infinity. Here x compactifies one side of Γ and y the other.

(4) LEMMA. *Let C be a simple closed curve in M.*

(i) *If M is a Klein bottle, C is two-sided and $M - C$ is connected, then $M - C$ is a cylinder;*

(ii) *If M is a torus with a cross-cap, C is two-sided and $M - C$ is connected, then $A_{xy}(M - C)$ is a real projective plane;*

(iii) *If M is a real projective plane and C is two-sided, then C bounds a unique disc;*

(iv) *If M is a torus with a cross-cap and C is one-sided, then $A_x(M - C)$ is either a torus or a Klein bottle.*

PROOF. We first prove (i). Let $\varphi\colon \mathbf{R} \times M \to M$ be a continuous flow on M with a finite number of fixed points (singularities) and such that C is transverse to φ. Let $A_{xy}(\varphi)$ be any of the continuous flows on $A_{xy}(M - \Gamma)$ induced by φ. Here, $A_{xy}(\varphi)$ has two singularities more than φ; namely x and y, both with index one. Thus, if we write $\chi(\ ) =$ Euler characteristic of ( ) and $S(\ ) =$ the sum of the indices of the singularities of ( ), we have $\chi(A_{xy}(M - C)) = S(A_{xy}(\varphi)) = 2 + S(\varphi) = 2 + \chi(M) = 2$, i.e., $A_{xy}(M - C)$ is a sphere; therefore, $M - C$ is a cylinder.

To prove (iii) we first note that if we suppose $M - C$ connected we get, as in the proof of (i), the following equality $\chi(A_{xy}(M - C)) = 2 + \chi(M - C)$, which implies the following contradiction $3 = 2 + \chi(M) = 2 + \chi(M - C) = \chi(A_{xy}(M - C)) \leqslant 2$. Let M_1 and M_2 be the connected components of $M - C$. Since M is not a sphere, both M_1 and M_2 cannot be discs. Now if we assume that neither M_1 nor M_2 is a disc, i.e. $\chi(A_x(M_1)) \leqslant 1$ and $\chi(A_y(M_2)) \leqslant 1$ we have $1 \geqslant \chi(A_x(M_1)) = 1 + \chi(M_1)$ and $1 \geqslant \chi(A_y(M_2)) = 1 + \chi(M_2)$, i.e., $\chi(M_1) \leqslant 0$ and $\chi(M_2) \leqslant 0$, which gives the contradiction $1 = \chi(M) = \chi(M_1) + \chi(M_2) \leqslant 0$.

(ii) and (iv) can be proved similarly to (i).

THEOREM 2. *Let φ be a continuous flow on the Klein bottle K^2. Every ω-recurrent (α-recurrent) trajectory γ of φ is trivial.*

PROOF. Let γ be an ω-recurrent trajectory of φ and C be a transverse circle to φ through γ.

Since there are arcs of the trajectory γ joining the two sides of C, without intersecting C, $K^2 - C$ is connected. We may assume (by Lemma (4)) that K^2 is an annulus where the boundary circles, C_1 and C_2, are oriented by the arrows as in Figure 2. They are identified by means of an orientation-preserving homeomorphism $h: C_1 \to C_2$.

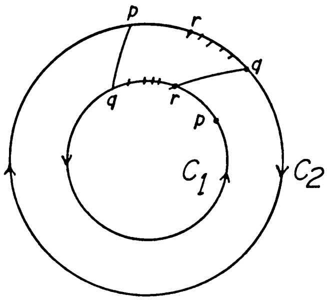

FIGURE 2

Let p, q and r be three consecutive points at which γ intersects C. Let pq be the arc of C with endpoints p and q, containing r. Since $M - C$ is an annulus, each trajectory passing through pq may intersect C again only at the segment $rq \subset pq$, with endpoints r and q (see Figure 2). It follows that the positive orbit of γ can accumulate only on points of rq and therefore not on p. This is a contradiction.

2. Proof of Theorem 1.

We denote by $\tilde{M}$ the torus with a cross-cap.

(1) LEMMA. *Let* φ: $\mathbf{R} \times \tilde{M} \to \tilde{M}$ *be a continuous flow on* $\tilde{M}$ *and* C *a transverse circle to* φ. *If* $\alpha = \overset{\frown}{p_1 p_2}$ *and* $\beta = \overset{\frown}{q_1 q_2}$ *are two distinct one-sided* (*two-sided*) C-*arcs of* φ, *then they determine an open quadrilateral contained in* $\tilde{M}$ *such that its edges* (*which may be nondisjoint*) *are* α, β *and two subarcs of* C *with endpoints* p_1, q_1 *and* p_2, q_2 *respectively*. (*See Figure* 3.)

PROOF. We observe that $\tilde{M} - C$ is connected; thus, $A_{xy}(\tilde{M} - C)$ is a real projective plane (Lemma 4, §1).

Certainly $\tilde{\alpha} = \alpha - C$ and $\tilde{\beta} = \beta - C$ are segments (of $A_{xy}(\tilde{M} - C)$) such that $\tilde{\alpha} \cup \tilde{\beta} \cup \{x\} \cup \{y\}$ is a two-sided simple closed curve bounding a unique open disc of $A_{xy}(\tilde{M} - C)$ (see Lemma 4, §1).

The lemma follows easily from this construction.

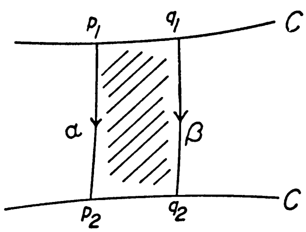

FIGURE 3

(2) PROPOSITION. *Let* φ: $\mathbf{R} \times \tilde{M} \to \tilde{M}$ *be a continuous flow on* $\tilde{M}$ *with some* ω*-recurrent trajectory* γ_1. *If* γ *is a trajectory of* φ *such that* $\omega(\gamma) \supset \gamma_1$ *and* G *is a transverse circle to* φ *through* γ_1, *then every* G*-arc of* γ, *with possible exception of one, is two-sided.*

PROOF. First we prove the following statement.

(a) If R is an α-invariant or ω-invariant (with respect to φ) Moebius band contained in M, then $\gamma_1 \cap R = \varnothing$.

If R is α-invariant and $p \in R \cap \gamma_1$, then the positive semitrajectory of γ_1, starting at p cannot leave R because otherwise it would not come back to R and therefore it would not accumulate at p. R α-invariant means that the negative semitrajectory of γ_1 starting at p is contained in R, that is $\gamma_1 \subset R$, which is a contradiction because γ_1, being a nontrivial ω-recurrent trajectory, cannot be contained in a Moebius band. A similar argument works when R is ω-invariant.

Let C be a transverse circle to φ. We are going to prove:

(b) There does not exist an open rectangle, say R, contained in M satisfying:

(b.1) two of the opposite edges of R, say q_0q_1 and q_1q_2 are segments of C such that one of them is contained in the other;

(b.2) the other two opposite edges of R are consecutive arcs, $\widehat{q_0q_1}$ and $\widehat{q_1q_2}$, of the same trajectory;

(b.3) the closure of R is an α-invariant or ω-invariant (with respect to φ) Moebius band.

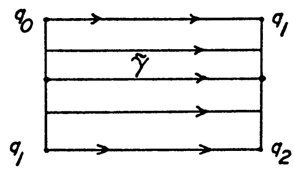

FIGURE 4

In fact, initially, we define a continuous flow $\tilde{\varphi}$: $\mathbf{R} \times M \to M$ satisfying: $\tilde{\varphi} = \varphi$ on the closure of $M - R$ and such that $\tilde{\varphi}$ has in the closure of R a one-sided closed trajectory $\tilde{\gamma}$ (see Figure 4). Certainly, by (a), γ_1 is also a nontrivial ω-recurrent trajectory of $\tilde{\varphi}$. We consider $A_x(M - \tilde{\gamma})$ and one of the flows on it, say $A_x(\tilde{\varphi})$, induced by $\tilde{\varphi}$. Certainly $\widehat{q_0q_1} \cap q_0q_1 = \{q_0, q_1\}$ and, since R is a Moebius band $(C - q_0q_1) \cup \widehat{q_0q_1}$ is a one-sided simple closed curve in $A_x(M - \tilde{\gamma})$. Thus $A_x(M - \tilde{\gamma})$ is a Klein bottle (see Lemma 4, §1) and $A_x(\tilde{\varphi})$ does not have nontrivial ω-recurrent trajectories. This contradiction proves (b).

Now, suppose that this proposition is false, then there is a transverse circle to φ, through γ_1, say C, such that:

(c.1) There is a sequence of consecutive C-arcs of γ: $A_j = \widehat{p_{j-1}p_j}$, $j = 1, 2, \ldots, m$, all of which are two-sided except A_1 and A_m.

(c.2) If $\tilde{C}$ is an arbitrary transverse circle to φ, through γ_1, and $B_1, B_2, \ldots, B_k$ is a sequence of consecutive $\tilde{C}$-arcs of γ all of which are two-sided, except B_1 and B_k, then $k \geqslant m$.

We will prove that $m \leqslant 3$. In fact, let us suppose that $m > 3$. Let $A_{m+1} = \widehat{p_m p_{m+1}}$ and $A_{m+2} = \widehat{p_{m+1}p_{m+2}}$ be the two consecutive C-arcs that follow A_m.

Since A_1 and A_m are one-sided (respectively A_2 and A_{m+1} are two-sided), according to Lemma 2, they are contained in the frontier of a unique open quadrilateral R_1 (resp. R_2). Let "$<$" refer to the orientation defined on C in such a way that the interval $I_0 = \{p | p_0 \leqslant p \leqslant p_{m-1}\}$ is an edge of R_1 (see Figure 5). Since the edges A_1 and A_m of R_1 reverse orientation, the opposite edge to I_0 is $I_1 = \{p | p_m \leqslant p \leqslant p_1\}$. We are going to show that I_1 is also an edge of R_2. To do this we first observe:

(d.1) $p_i \notin I_0$ for $0 < i < m - 1$ and $i = m, m + 1$ (otherwise $A_{i+1} \subset R_1$ would be one-sided, and m is not minimal, so (c.2) fails).

(d.2) $p_i \notin I_1$ for $1 < i < m$ and $i = 0$ (otherwise $A_i \subset R_1$).

(d.3) If $I_0 \cap I_1 \neq \varnothing$ then $p_0 \in I_1$, $p_{m-1} \in I_1$, $p_m \in I_0$ or $p_1 \in I_0$, contradicting (d.1) or (d.2).

By (d.3), $I_0 \subset C - I_1$; besides, A_1 and A_m, being one-sided, cannot belong to R_2. Therefore, I_1 is an edge of R_2. It then follows, since R_2 has two-sided edges, that the other edge of R_2 is $I_2 = \{p | p_{m+1} \leqslant p \leqslant p_2\}$.

In the following sequence of observations we are going to prove that I_0, I_1 and I_2 are pairwise disjoint.

(e.1) If $I_1 \cap I_2 \neq \varnothing$ then $p_{m+1} \in I_1$, $p_2 \in I_1$, $p_1 \in I_2$ or $p_m \in I_2$, which implies $p_m \in I_0$, $p_1 \in I_0$, $p_0 \in I_1$ or $p_{m-1} \in I_1$, respectively, contradicting (d.1) or (d.2).

(e.2) $p_{m-1} \notin I_2$, otherwise $p_{m-3} \in I_0$ $(m > 3)$ contradicts (c.2) because m would not be minimal.

(e.3) $p_0 \notin I_2$ (the same argument as (e.2)).

(e.4) If $I_0 \cap I_2 \neq \varnothing$, then $p_2 \in I_0$, $p_{m+1} \in I_0$, $p_{m-1} \in I_2$ or $p_0 \in I_2$ which contradicts (d.1), (e.2) or (e.3).

In this situation, considering the possible orderings of the points $p_1, p_2, p_{m-1}, p_m, p_{m+1}$ in the oriented arc $C - \{p_0\}$, we have two cases:

(f.1) $p_{m-1} < p_{m+1} < p_2 < p_m < p_1$ (Figure 5).

(f.2) $p_{m-1} < p_m < p_1 < p_{m+1} < p_2$.

We consider only case (f.1). By using a flow box centered at A_2 we get a point $\tilde{p}_2$ near p_2 and a cross section θ (to the flow φ) with endpoints p_1 and $\tilde{p}_2$, in such a way that $\theta \cap \widehat{p_0 p_{m+1}} = \{p_1\}$ (Figure 5 shows θ as a broken segment).

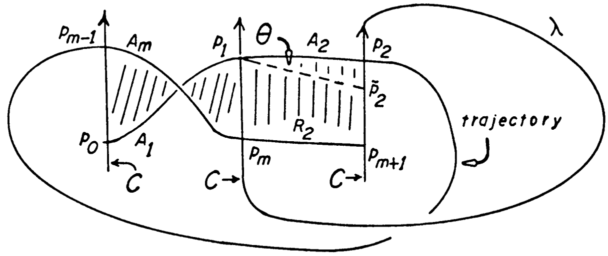

FIGURE 5

Let λ be the subarc of $C - \{p_0\}$ with endpoints $\tilde{p}_2$ and p_1 (see Figure 5). Certainly $\tilde{C} = (C - \lambda) \cup \theta$ is a transverse circle to φ and $\widehat{p_0 p_1}$ and $\widehat{p_{m-1} p_{m+1}}$ are one-sided $\tilde{C}$-arcs. Moreover, the number of C-arcs determined by $\widehat{p_0 p_{m+1}}$ is less than m; this is in contradiction with (c.2) because, as we will see, $\gamma_1 \cap \tilde{C} \neq \varnothing$. In fact, it is easy to verify that $M - \tilde{C}$ is connected and if we assumed $\gamma_1 \cap \tilde{C} = \varnothing$, by considering the projective plane $A_{xy}(M - \tilde{C})$, we would get a flow on it, say $A_{xy}(\varphi)$, induced by φ, with the nontrivial ω-recurrent trajectory γ_1. This proves that $m \leqslant 3$.

If we assume $m = 2$, there is a quadrilateral R whose edges are the one-sided C-arcs of γ: $A_1 = \widehat{p_1 p_2}$, $A_2 = \widehat{p_2 p_3}$ and the two subarcs of C that we denote by $\overline{p_1 p_2}$ and $\overline{p_2 p_3}$. The closure of R is a Moebius band having properties which contradict (b). In a similar way if we assume $m = 3$ we get a contradiction with (b). This proves this proposition.

(3) DEFINITION. $X \in \mathfrak{X}^r(M)$, $r \geqslant 1$, is a Morse-Smale vector field if:

(i) There is only a finite number of singularities, all hyperbolic.

(ii) The α- and ω-limit sets of every trajectory can only be singularities or closed orbits.

(iii) No trajectory connects saddle points.

(iv) There is only a finite number of closed orbits, all hyperbolic.

We denote by Σ^r the set of Morse-Smale vector fields.

(4) DEFINITION. $X \in \mathfrak{X}^r(M)$, $r \geqslant 1$, is said to be structurally stable if there is a neighborhood $\mathcal{V}$ of X such that whenever $Y \in \mathcal{V}$ there is a homeomorphism of M onto itself transforming trajectories of X onto trajectories of Y.

THEOREM 1. *Let M be the torus with a cross-cap. $X \in \mathfrak{X}^r(M)$ is structurally stable if and only if $X \in \Sigma^r$. Moreover, Σ^r is open and dense in $\mathfrak{X}^r(M)$.*

PROOF. Our proof is the same as given by Peixoto in [2].

It is possible to verify that to extend Peixoto's proof of his theorem to nonorientable manifolds it is sufficient to extend to them the following lemma (see [2, Lemma 5, p. 100]).

(*) Let p be a point of a nontrivial ω-recurrent trajectory of $X \in \mathfrak{X}^r(M)$, where M is an orientable two-manifold and X has a finite number of singularities, all hyperbolic. Assume that there exist two trajectories λ_1 and λ_2 of X such that $\alpha(\lambda_1)$ and $\omega(\lambda_2)$ are saddle points and that furthermore $p \in \omega(\lambda_1) \cap \alpha(\lambda_2)$. Then X can be approximated by a system having one more saddle connection than X has.

When M is an orientable manifold or a torus with a cross-cap, (*) is proved as follows.

Let C be a transverse circle to X, passing through p, disjoint from the saddle connections. Let X_t, $t \in \mathbf{R}$, denote the flow induced by X and d denote a metric on $X_1(C) = C_1$. Let "$<$" be an orientation defined on C_1. We are going to write $p < q, p, q \in C_1$, when the segment $\{x \in C_1 | p \leqslant x \leqslant q\}$ has smaller length than the segment $\{x \in C_1 | q \leqslant x \leqslant p\}$. Given a small $\varepsilon > 0$ we can find a vector field $Y \in \mathfrak{X}^r(M)$ close to 0 in $\mathfrak{X}^r(M)$, transversal to X in $\{\bigcup X_t(C) | -1 < t < 1\}$ and equal to 0 outside this set. Certainly there exists $\delta > 0$ such that if we denote by $X_t^{(s)}$, $t \in \mathbf{R}$, the flow induced by $X^{(s)} = X + sY, s \in [-1, 1]$, we have:

(a) $\forall p \in C_{-1} = X_{-1}(C), \sigma \in \{-1, 1\}, d(X_2^{(\sigma)}(p), X_2(p)) \geqslant \delta.$

Moreover "$<$" can be chosen in order to get

(b) $\forall p \in C_{-1}, X_2^{(-1)}(p) < X_2(p) < X_2^{(1)}(p).$

Let $\lambda_1(s)$ and $\lambda_2(s)$ be the saddle separatrices of $X^{(s)}$ satisfying $\alpha(\lambda_1(s)) = \alpha(\lambda_1)$ and $\omega(\lambda_2(s)) = \omega(\lambda_2)$. Let $p_m(s)$ (resp. $q_n(s)$), $n = 1, 2, \ldots,$ be the nth point where the trajectory $\lambda_1(s)$ (resp. $\lambda_2(s)$) intersects positively (resp. negatively) C_1. Now we state the following fact which is an easy consequence of [2, Lemma 3]. If there exists $s \in [-1, 1]$ such that $\lambda_1(s)$ (resp. $\lambda_2(s)$) intersects C in a finite number of points, then there exists $s_0 \in [-1, s]$ such that $\lambda_1(s_0)$ (resp. $\lambda_2(s_0)$) is a saddle connection.

Let us suppose that $\forall s \in [-1, 1]$, neither $\lambda_1(s)$ nor $\lambda_2(s)$ is a saddle connection. Since p is a limit point of $\{p_m(0)\}$ and $\{q_l(0)\}$ there exist $N, L \in \mathbf{N}$ such that $d(p_N(0), q_L(0)) < \delta$. We are going to consider only the case $p_N < q_L$.

According to Proposition 2 and the construction of a transverse circle to X (see Lemma 2, §1) we can choose C in such a way that

(c) All the C-arcs of λ_1 and λ_2 are two-sided.

In consequence,

(d) When M is orientable or is a torus with a cross-cap, we have that if s increases then $p_N(s)$ increases and $q_L(s)$ decreases.

Thus, (a), (b) and (d) imply that there exists $\theta \in [0, 1]$ such that $p_N(\theta) = q_L(\theta)$, i.e. $\lambda_1(\theta)$ is a saddle connection.

We remark that it is not yet known if (c), consequently (d), holds for a nonorientable two-manifold. In [2] Peixoto observed this fact, however, indirectly he assumed (c) true for any two manifold.

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